LOWER BOUNDS AT INFINITY FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

BY

LARS HÖRMANDER

ABSTRACT

Two extensions of a classical theorem of Rellich are proved: (1) Let $P = P(-i\partial/\partial x)$ be a partial differential operator with constant coefficients in $\mathbb{R}^n$, let the manifolds contained in $\{\xi \in \mathbb{R}^n;\ P(\xi) = 0\}$ have codimension $\geqq k > 0$, and denote by Γ an open cone in $\mathbb{R}^n$ intersecting each normal plane of every such manifold. If $u \in \mathcal{S}' \cap L^2_{\text{loc}}$, $Pu = 0$ and

$$\varliminf_{R\to\infty} R^{-k} \int_{\Gamma_R} |u(x)|^2 dx = 0, \qquad \Gamma_R = \{\xi \in \Gamma,\ R < |\xi| < 2R\}$$

it follows that $u = 0$. (2) Assume in addition that each irreducibe lfactor of P vanishes on a real hypersurface and that Γ contains both normal directions at some such point. If $u \in \mathcal{S}' \cap L^2_{\text{loc}}$ and $P(D)\,u$ has compact support, the same condition with $k = 1$ implies that u has compact support. In both results the hypotheses on the cone Γ and on the operator P are minimal.

1. Introduction

According to a classical theorem of Rellich [8] a solution of the reduced wave equation $\Delta u + u = 0$ outside a ball in $\mathbb{R}^n$ must vanish identically if $u(x)|x|^{(n-1)/2} \to 0$ as $x \to \infty$. Extensions of this result have been given for large classes of operators (see [1, 5, 6, 9] and further references in these papers). Our aim here is to complete the study of the constant coefficient case in Littman [5, 6].

If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $P(D)u = 0$ where P is a polynomial in $D = -i\partial/\partial x$, then the support of the Fourier transform $\hat{u}$ is contained in the set A of real zeros

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of P. Since A is the union of analytic manifolds we begin the study of the asymptotic behavior of u in Section 2 by examining the Fourier transform of a distribution v supported by a smooth manifold in $\mathbb{R}^n$. When v is tangentially smooth it is known that the wave front set is contained in the normal bundle of the manifold at the support of v (see [3, Sect. 2.5]). It is therefore natural that the normal directions play a crucial role in the statements proved in Section 2. Disregarding such precise information here we may sum up the results obtained as follows: If A is non-empty and of codimension k, then the equation $P(D)u = 0$ has solutions $u \in C^\infty$ such that

$$0 < \int_{|x|<R} |u(x)|^2 dx \leqq CR^k, \qquad R > 0. \tag{1.1}$$

On the other hand, if $u \in \mathscr{S}' \cap L^2_{\mathrm{loc}}$ satisfies the equation $P(D)u = 0$ and

$$\lim_{R\to\infty} R^{-k} \int_{R/2<|x|<R} |u(x)|^2 dx = 0, \tag{1.2}$$

then $u = 0$.

Section 3 is devoted to the equation $P(D)u = f$ where f is a distribution of compact support. First we give quite precise conditions on the decrease of u which guarantee that the equation has another solution of compact support. Subtraction of this solution and application of the results proved in Section 2 gives extensions of Rellich's theorem where the hypotheses are proved to be essentially minimal. In particular it follows that $u \in L^2$, $P(D)u \in \mathscr{E}'$ implies $u \in \mathscr{E}'$ if and only if every irreducible factor of P vanishes on a real hypersurface. This property was established by Agmon [1] for some classes of operators with variable coefficients.

2. Fourier transforms of distributions supported by manifolds and analytic sets

Our first result is elementary and essentially well known (see [5]):

THEOREM 2.1. *If u is a smooth density with compact support on a C^∞ submanifold M of $\mathbb{R}^n$ of codimension k, then*

$$\int_{|\xi|<R} |\hat{u}(\xi)|^2 d\xi \leqq CR^k, \quad R > 0. \tag{2.1}$$

If Γ is a closed cone in $\mathbb{R}^n$ which contains no element $\neq 0$ which is normal to M at a point in supp u, *the*

$$(2.2)\qquad |\hat{u}(\xi)| \leqq C_N(1+|\xi|)^{-N}, \qquad \xi \in \Gamma,$$

for every integer N.

PROOF. By a partition of unity the proof is reduced to the case where M is of the form

$$x'' = \phi(x');\; x' = (x_1,\cdots,x_{n-k}) \in \omega \subset \mathbb{R}^{n-k},\; x'' = (x_{n-k+1},\cdots,x_n).$$

Here ω is open and $\phi \in C^\infty(\omega)$. We can write $u = a(x')dx'$ where $a \in C_0^\infty(\omega)$ and obtain

$$\hat{u}(\xi) = \int \exp(-i(\langle x',\xi'\rangle + \langle \phi(x'),\xi''\rangle))a(x')dx'.$$

If $x' \in \operatorname{supp} a$ is a critical point of the function in the exponent, then

$$\langle t,\xi'\rangle + \langle \phi'(x')t,\xi''\rangle = 0, \qquad t \in \mathbb{R}^{n-k},$$

which means that $\xi = (\xi',\xi'')$ is a normal of M at $(x',\phi(x'))$. Hence there is no critical point when $0 \neq \xi \in \Gamma$ which gives (2.2) by repeated partial integrations, By Parseval's formula

$$\int\int_{|\xi''|<R} |\hat{u}(\xi)|^2 d\xi'\, d\xi'' = (2\pi)^{n-k}\int |a(x')|^2 dx' \int_{|\xi''|<R} d\xi'' = CR^k$$

which implies (2.1).

In spite of the simple proof the estimate (2.1) is optimal:

THEOREM 2.2. *Let $u \in \mathscr{S}'$, $\hat{u} \in L^2_{\text{loc}}$, and assume that there is a point $x_0 \in \operatorname{supp} u$ such that* supp u *in a neighborhood of x_0 is contained in a C^∞ manifold M of codimension k. If $\theta \in \mathbb{R}^n$ is a normal of M at x_0 and if $\varepsilon > 0$, then*

$$(2.3)\qquad \varliminf_{R\to\infty} R^{-k} \int_{|\xi/R-\theta|<\varepsilon} |\hat{u}(\xi)|^2 d\xi > 0.$$

We shall prepare for the proof by a lemma which also shows that the hypothesis $\hat{u} \in L^2_{\text{loc}}$ made in the theorem is not restrictive.

LEMMA 2.3. *Let $u \in \mathscr{S}'(\mathbb{R}^n)$, $\hat{u} \in L^2_{\text{loc}}$, $\theta \in \mathbb{R}^n$ and $\varepsilon > 0$. If $\chi \in C_0^\infty(\mathbb{R}^n)$ and $v = \chi u$ it follows that for every $k \in \mathbb{R}$*

$$(2.4)\qquad \varliminf_{R\to\infty} R^{-k}\int_{|\xi/R-\theta|<\varepsilon} |\hat{v}(\xi)|^2 d\xi \leqq C \varliminf_{R\to\infty} R^{-k}\int_{|\xi/R-\theta|<2\varepsilon} |\hat{u}(\xi)|^2 d\xi,$$

where $C = (2\pi)^{-n} \int |\hat{\chi}|\,d\xi$.

PROOF. Choose $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\psi(\xi) = 1$ when $|\xi| < \varepsilon/2$ and $\psi(\xi) = 0$ when $|\xi| > \varepsilon$, and set $\psi_R(\xi) = \psi(\xi/R)$. Then

$$\hat{v} = (2\pi)^{-n}\hat{\chi} * \hat{u} = (2\pi)^{-n}((\psi_R\hat{\chi}) * \hat{u} + ((1-\psi_R)\hat{\chi}) * \hat{u}),$$

$$\int_{|\xi/R-\theta|<\varepsilon} |(\psi_R\hat{\chi}) * \hat{u}|^2 d\xi \leqq \left(\int |\hat{\chi}|\, d\xi\right)^2 \int_{|\xi/R-\theta|<2\varepsilon} |\hat{u}|^2 d\xi.$$

For arbitrary multi-indices α and β we have for any N when $|\xi/R-\theta|<\varepsilon$

$$\sup_\eta |\eta^\alpha D_\eta^\beta(1-\psi_R(\xi-\eta))\hat{\chi}(\xi-\eta)| \leqq C_N R^{-N}. \tag{2.5}$$

In fact, the function to estimate vanishes unless $|\xi-\eta| > \varepsilon R/2$, and since $|\eta| \leqq |\xi| + |\xi-\eta| \leqq R(|\theta|+\varepsilon) + |\xi-\eta|$ we obtain (2.5) in view of the rapid decrease of $\hat{\chi}$. Since $\hat{u} \in \mathscr{S}'$ we conclude that

$$|((1-\psi_R)\hat{\chi}) * \hat{u}(\xi)| \leqq C_N R^{-N} \text{ if } |\xi/R-\theta|<\varepsilon,$$

which proves the lemma.

Proof of Theorem 2.2. In view of Lemma 2.3 it is no restriction to assume that $\operatorname{supp} u$ is a compact subset of M. We may also assume that $x_0 = 0$ and, as in the proof of Theorem 2.1, that M is defined by $x'' = \phi(x')$ where $x' \in \omega \subset \mathbb{R}^{n-k}$, $\phi \in C^\infty(\omega)$ and $\phi(0) = \phi'(0) = 0$. The normal θ is then of the form $(0, \theta'')$.

Choose a conic neighborhood V of $\{(0,\xi''); \xi'' \in \mathbb{R}^k\backslash 0\}$ so that

$$|\xi-\theta| < \varepsilon \text{ if } \xi = (\xi',\xi'') \in V \text{ and } |\xi''-\theta''| < \varepsilon/2. \tag{2.6}$$

When $\chi \in C_0^\infty(\omega)$ it follows from Theorem 2.1 that

$$I_\chi(\xi) = \int \chi(x') \exp(i(\langle x',\xi'\rangle + \langle\phi(x'),\xi''\rangle) dx'$$

is rapidly decreasing outside V if $\operatorname{supp}\chi$ is sufficiently close to 0. Choose $\psi \in C_0^\infty(\mathbb{R}^k)$ so that $|\xi''-\theta''| < \varepsilon/2$ for $\xi'' \in \operatorname{supp}\psi$, and set

$$f_R(x) = \chi(x')\hat{\psi}(R(x''-\phi(x'))).$$

By Fourier's inversion formula

$$\hat{f}_R(\xi) = (2\pi)^k R^{-k} \psi(-\xi''/R) I_\chi(-\xi).$$

Hence

$$u(f_R) = (2\pi)^{-n} \int \hat{u}(\xi)\hat{f}_R(-\xi) d\xi = (2\pi)^{k-n} R^{-k} \int \psi(\xi''/R) I_\chi(\xi)\hat{u}(\xi) d\xi.$$

It follows from (2.6) that $|\xi/R-\theta| < \varepsilon$ in the intersection of V and the support of the integrand. Thus we obtain using (2.1) and Cauchy-Schwarz' inequality

$$|u(f_R)| \leqq C((R^{-k} \int_{|\xi/R-\theta|<\varepsilon} |\hat{u}(\xi)|^2 d\xi)^{1/2} + CR^{-k} \int_{CV} |I_\chi(\xi)\hat{u}(\xi)|\, d\xi.$$

The last integral is convergent by Theorem 2.1 so if (2.3) is not valid we conclude that

$$\lim_{R\to\infty} |u(f_R)| = 0. \tag{2.7}$$

Let v be the composition of u and the map $x \to (x', x'' + \phi(x'))$ (defined arbitrarily for $x' \notin \omega$). Then the support of v is in the plane $x'' = 0$ and we can write v as a finite sum

$$v = \sum v_\alpha \otimes D^\alpha \delta$$

where $v_\alpha \in \mathscr{E}'(\mathbb{R}^{n-k})$, δ is the Dirac measure at 0 in $\mathbb{R}^k$ and $\alpha = (\alpha_{n-k+1}, \cdots, \alpha_n)$. We can then write (2.7) in the form

$$\lim_{R\to\infty} \left|\sum v_\alpha(\chi) R^{|\alpha|}(-D)^\alpha \hat{\psi}(0)\right| = 0. \tag{2.8}$$

Since ψ can be chosen so that the occurring derivatives of $\hat{\psi}$ have prescribed values, we conclude that $v_\alpha(\chi) = 0$ for all $\chi \in C_0^\infty(\omega)$ with support close to 0. Hence $0 \notin \operatorname{supp} v_j$ so $0 \notin \operatorname{supp} u$ and the theorem is proved.

REMARK. The proof shows that the limit in (2.3) is $+\infty$ unless u has an L^2 density.

Before stating the next theorem we recall that a subset A of $\mathbb{R}^n$ is called real analytic if it can be defined locally by analytic equations. The codimension is the minimal codimension of analytic manifolds M with $M \subset A$.

THEOREM 2.4. *Let $u \in \mathscr{S}'(\mathbb{R}^n)$ be supported by a real analytic set A of codimension $k > 0$, and assume that $\hat{u} \in L^2_{\mathrm{loc}}$. Set*

$$\Gamma_R = \{\xi \in \Gamma;\, R < |\xi| < 2R\}$$

where Γ is an open cone in $\mathbb{R}^n$ which for every analytic manifold $M \subset A$ and $x_0 \in M$ contains some normal of M at x_0. If

$$\lim_{R\to\infty} R^{-k} \int_{\Gamma_R} |\hat{u}(\xi)|^2 d\xi = 0$$

it follows that $u = 0$.

PROOF. Let $x_0 \in A$ and assume that $A_1 \subset A$ is an analytic set in a neighborhood Ω of x_0 such that $\Omega \cap \operatorname{supp} u \subset A_1$. Let d be the codimension of A_1. Since $d \geqq k$ it follows from Theorem 2.2 that no regular point of A_1 of dimension $n - d$ is in $\operatorname{supp} u$. But the other points are contained in an analytic set A_2 of codimension

$> d$ (see [7, Chap. III, Th. 1 and Prop. 7]). By repeating the argument we conclude that $x_0 \notin \operatorname{supp} u$.

REMARK. We could use different cones for different x_0 in Theorem 2.4 as well as in the results below.

The analyticity assumption is no restriction in the application to differential equations:

COROLLARY 2.5. *Let $u \in \mathcal{S}' \cap L^2_{\text{loc}}$ satisfy a differential equation $P(D)u = 0$ with constant coefficients. Assume that $A = \{\xi \in \mathbb{R}^n, \ P(\xi) = 0\}$ is not empty and of codimension k, and let Γ, Γ_R be as in Theorem 2.4. If*

$$\varliminf_{R \to \infty} R^{-k} \int_{\Gamma_R} |u(x)|^2 dx = 0$$

it follows that $u = 0$.

The following theorem shows that Corollary 2.5 is very precise.

THEOREM 2.6. *Assume that Γ is an open cone in $\mathbb{R}^n$ and N an integer such that every $u \in \mathcal{S}' \cap C^\infty$ with $P(D)u = 0$ and*

$$\varliminf_{R \to \infty} R^{-N} \int_{\Gamma_R} |u(x)|^2 dx = 0 \tag{2.9}$$

is equal to 0. If $M \subset \mathbb{R}^n$ is a C^∞ manifold where P vanishes and if $x_0 \in M$, it follows that the closure of Γ contains some normal $\neq 0$ of M at x_0 and that $N \leqq \operatorname{codim} M$.

PROOF. Let $\hat{u}$ be a C_0^∞ density on M. Then $Pu = 0$ and (2.9) follows from (2.1) if $\operatorname{codim} M < N$ which is therefore impossible. If $\bar{\Gamma}$ contains no normal $\neq 0$ of M at x_0 and if supp u is sufficiently close to x_0 the condition (2.9) follows from (2.2) for any N which completes the proof.

3. Compact perturbations

We shall now pass to the study of solutions of the inhomogeneous equation $P(D)u = f$ when $f \in \mathcal{E}'$. Since the study of u at infinity can be reduced to Corollary 2.5 if there is another solution of compact support we shall first examine when such a solution exists.

THEOREM 3.1. *Let $u \in \mathcal{S}' \cap L^2_{\text{loc}}$ and assume that $P(D)u = f \in \mathcal{E}'$. Assume further that $P = c\, P_1^{m_1} \cdots P_k^{m_k}$ where c is a constant and for every j*

$$P_j \text{ is real and irreducible} \tag{3.1}$$

(3.2) $P_j(\xi^j) = 0$ *and* $N^j = \operatorname{grad} P_j(\xi^j) \neq 0$ *for some* $\xi^j \in \mathbb{R}^n$.

Let Γ *be an open cone containing* N^j *and* $-N^j$ *for every* j, *and set* $\Gamma_R = \{x \in \Gamma, R < |x| < 2R\}$. *If*

$$\varliminf_{R\to\infty} R^{-1} \int_{\Gamma_R} |u(x)|^2 dx = 0, \tag{3.3}$$

it follows that $P(D)v = f$ *for some* $v \in \mathscr{E}' \cap L^2$.

PROOF. It suffices to show that $\hat{f}/P$ is an entire function (see [2, Th. 3.4.2 and 3.2.2]). Since we can assume that the irreducible factors P_j are different this follows if we show that $\hat{f}$ vanishes of order m_j at the real zeros of P_j near ξ^j. With suitable coordinates these are of the form

$$\xi_n = s(\xi'),\ \xi' \in \omega \subset \mathbb{R}^{n-1},$$

where $s \in C^\infty(\omega)$ is real valued, $\xi^j = (\xi_0', s(\xi_0'))$ and N^j is proportional to $(s'(\xi_0'), -1)$.

Our argument is similar to the proof of Theorem 2.2. Choosing $\chi \in C_0^\infty(\omega)$ with support close to ξ_0' and $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ in a neighborhood of 0, we set with $0 \leqq \nu < m_j$

$$g_R(\xi) = \chi(\xi')R^{\nu+1}\, \hat{\psi}^{(\nu)}(R(s(\xi') - \xi_n)).$$

If $F_\nu = \partial^\nu \hat{f}/\partial\xi_n^\nu$ we obtain by dominated convergence

$$\begin{aligned}\hat{f}(g_R) &= \iint \chi(\xi')R^{\nu+1}\hat{\psi}^{(\nu)}(R(s(\xi') - \xi_n))\hat{f}(\xi', \xi_n)d\xi' d\xi_n \\ &= \iint \chi(\xi')\hat{\psi}(\tau)F_\nu(\xi', s(\xi') - \tau/R)d\tau d\xi' \to 2\pi \int \chi(\xi')F_\nu(\xi', s(\xi'))d\xi',\end{aligned}$$

for $\int \hat{\psi}(\tau)d\tau = 2\pi$. The proof will therefore be complete if we show that $\varliminf_{R\to\infty} |\hat{f}(g_R)| = 0$ when $\operatorname{supp}\chi$ is sufficiently close to ξ_0'.

Since $\hat{f} = P\hat{u}$ we have $\hat{f}(g_R) = \hat{u}(h_R)$ where by Taylor's formula

$$h_R(\xi) = P(\xi)g_R(\xi) = R^{\nu+1}\hat{\psi}^{(\nu)}(R(s(\xi') - \xi_n)) \sum_{m_j}^{m} (\xi_n - s(\xi'))^\mu a_\mu(\xi').$$

Here $a_\mu \in C_0^\infty(\omega)$ and $\operatorname{supp} a_\mu \subset \operatorname{supp}\chi$. The Fourier transform of $t^\mu \hat{\psi}^{(\nu)}(-t)$ is $(2\pi)(i\partial/\tau)^\mu(-i\tau)^\nu\psi(\tau) = \psi_{\nu\mu}(\tau) \in C_0^\infty(\mathbb{R})$. Since $\mu \geqq m_j > \nu$ we have $\operatorname{supp}\psi_{\nu\mu} \subset \operatorname{supp}\psi'$ which does not contain the origin. With

$$I_\mu(x) = \int \exp(-i(\langle x', \xi'\rangle + x_n s(\xi'))a_\mu(\xi')d\xi'$$

we obtain as in the proof of Theorem 2.2.

$$(3.4) \qquad \hat{h}_R(x) = \sum_{mj}^{m} R^{\nu-\mu}\, \psi_{\nu\mu}(x_n/R) I_\mu(x).$$

Define $v \in \mathscr{S}' \cap C^\infty$ by $\hat{v} = \rho\hat{u}$ where $\rho \in C_0^\infty$ is equal to 1 in a neighborhood of ξ^j. It follows from Lemma 2.3 that (3.3) remains valid if $\hat{u}$ is replaced by by $\hat{v}$ and Γ is replaced by another conic neighborhood of $\pm N^j$. If supp χ is sufficiently close to ξ_0' we have $\hat{v}(h_R) - \hat{u}(h_R) \to 0$ as $R \to \infty$. Since $v(x) = O(|x|^N)$ for some N and since $\nu - \mu \leqq -1$ in (3.4), the proof of (2.7) gives $\underline{\lim}_{R\to\infty} |v(\hat{h}_R)| = 0$. Hence $\underline{\lim} |\hat{u}(h_R)| = 0$ which completes the proof.

Corollary 3.2. *Let P satisfy the hypotheses of Theorem* 3.1 *and assume that the open cone* Γ *has the properties required in Theorem* 3.1 *as well as in Corollary* 2.5. *If* $P(D)u = f \in \mathscr{E}'$, $u \in \mathscr{S}' \cap L^2_{loc}$ *and* (3.3) *is valid, it follows that* $u \in \mathscr{E}'$.

Proof. By Theorem 3.1 we have $P(D)u = P(D)v$ for some $v \in \mathscr{E}' \cap L^2$, and from Corollary 2.5 it follows that $u - v = 0$.

Note that supp$f$ and supp$u$ have the same convex hulls by the theorem of supports (see [2, Lemma 3.4.3]).

The following consequence was pointed out by I. Segal.

Corollary 3.3. *If P satisfies the hypotheses of Theorem* 3.1 *and* $u \in L^q(\mathbb{R}^n)$, $P(D)u \in \mathscr{E}'$, *it follows that u has compact support if* $q \leqq 2n/(n-1)$.

Proof. If $\chi \in C_0^\infty$ and $v = u * \chi$ we have $v \in L^r$ for $r \geqq q$ and $P(D)v \in \mathscr{E}'$. If $r > 2$ then

$$\int_{R<|x|<2R} |v(x)|^2 dx \leqq \left(\int_{R<|x|<2R} |v(x)|^r dx\right)^{2/r} \left(\int_{R<|x|<2R} dx\right)^{1-2/r} = o(R)$$

if $n(1-2/r) \leqq 1$, that is, $r \leqq 2n/(n-1)$. Hence v has compact support by Corollary 3.2, and supp v is contained in the convex hull of the support of $\chi * P(D)u$. When χ tends to δ the corollary follows.

Note that the statement is false for $q > 2n/(n-1)$ if $P = \Delta + 1$.

Corollary 3.2 is our extension of Rellich's theorem. The rest of the paper is devoted to a proof that the hypotheses made cannot be much relaxed.

Theorem 3.4. *Assume that P has an irreducible factor p which is not proportional to a real polynomial or has no simple real zero. For any integer N one can then find* $u \in L^\infty \cap C^\infty$ *so that* $P(D)u = f \in \mathscr{E}'$ *and* $u(x) = o(|x|^{-N})$ *but* $u \notin \mathscr{E}'$.

For the proof we need a lemma.

LEMMA 3.5. *Let p be an irreducible polynomial. If p is not proportional to a real polynomial or if p has no simple real zero there is a polynomial q relatively prime to p such that for some positive constants C, M_1, M_2*

$$|q(\xi)|^{M_1} \leqq C|p(\xi)|(1+|\xi|)^{M_2},\ \xi \in \mathbb{R}^n. \tag{3.5}$$

PROOF. First note that one can find q relatively prime to p so that $q = 0$ at every real zero of p. In fact, if p is not proportional to a real polynomial we can choose $q = \bar{p}$ and if p has no simple real zeros we can take $q = \partial p/\partial\xi_j$ with j chosen so that $\partial p/\partial \xi_j$ is not identically 0.

To prove that (3.5) is valid for such polynomials q we first assume that $|\xi| \leqq 1$. If p has no zero with $|\xi| \leqq 1$ there is nothing to prove so assuming that such zeros exist we set for $t > 0$

$$f(t) = \sup\{|q(\xi)|;\ |\xi| \leqq 1,\ |p(\xi)| \leqq t\}.$$

This increasing function of t is piecewise algebraic by the Tarski-Seidenberg theorem (see [2, Appendix]). Since q is chosen so that $f(t) \to 0$ when $t \to 0$ the Puiseux series expansion of f shows that $f(t) \leqq Ct^\gamma$ for some C and $\gamma > 0$. Hence $|q(\xi)| \leqq C|p(\xi)|^\gamma$ when $|\xi| \leqq 1$ which proves (3.5) then. If we introduce $\eta = \xi/|\xi|^2$ as a new variable and remove denominators we can make the same conclusion when $|\xi| > 1$.

PROOF OF THEOREM 3.4. With q chosen according to Lemma 3.5 and a large integer M we set

$$\hat{u}(\xi) = q(\xi)^M p(\xi)^{-1}\hat{g}(\xi).$$

Here $g \in C_0^\infty$ and $\hat{g}(\xi) \neq 0$ for some complex ξ with $p(\xi) = 0$ and $q(\xi) \neq 0$. Then $\hat{u}$ is not entire so u does not have compact support. If $M - N > M_1(1+N)$ it follows from (3.5) that $\hat{u} \in C^N$ and that the derivatives of order $\leqq N$ are rapidly decreasing. Hence $D^\alpha u(x) = o(|x|^{-N})$ for every α. Now

$$P(\xi)\hat{u}(\xi) = q(\xi)^M(P(\xi)/p(\xi))\hat{g}(\xi) = R(\xi)\hat{g}(\xi)$$

where R is a polynomial. It follows that $P(D)u = R(D)g \in C_0^\infty$ which proves the theorem.

REMARK. Trèves [9] has proved that

$$P(D)u = f \in \mathscr{E}',\ u(x) = o(|x|^{-N})\,\forall N \Rightarrow u \in \mathscr{E}'$$

if and only if every irreducible factor of P has some real zero. By Theorem 3.3 this is not true for any fixed N unless every factor has real zeros of codimension one, and in that case Corollary 3.2 shows that it suffices to take $N = (n-1)/2$ as in the original Rellich theorem. Thus there is a fundamental difference between the case of real zeros of codimension one and real zeros of higher codimension in the problems considered in this section but not in those discussed in Section 2.

We shall finally examine the necessity of the condition on Γ in Corollary 3.2. By Theorem 2.6 we know already that $\bar{\Gamma}$ must contain either $\operatorname{grad} P_j$ or $-\operatorname{grad} P_j$ for every zero of P_j so we include this in the hypotheses.

THEOREM 3.6. *Let p be an irreducible real polynomial and Γ a closed semi-algebraic cone in $\mathbb{R}^n$ such that $\operatorname{grad} p(\xi)$ or $-\operatorname{grad} p(\xi)$ is in Γ for every real zero of p. Assume that there is no real ξ with $p(\xi) = 0$ and $\operatorname{grad} p(\xi) \neq 0$ such that $\operatorname{grad} p(\xi)$ and $-\operatorname{grad} p(\xi)$ are both in Γ. Then one can for every integer N find $u \in \mathscr{S}' \cap C^\infty$ such that $p(D)u = f \in \mathscr{E}'$ and $u(x) = o(|x|^{-N})$ in Γ but $u \notin \mathscr{E}'$.*

The statement is of course contained in Theorem 3.4 if p has no simple real zeros so we assume that such zeros exist. If p is a factor of P we can replace p by P in the conclusion by simply applying the differential operator $P(D)/p(D)$.

The first step in the proof is to make the hypotheses quantitative by means of the Tarski-Seidenberg theorem.

LEMMA 3.7. *When the hypotheses of Theorem 3.6 are fulfilled there are constants C, M_1, M_2 such that when $\xi \in \mathbb{R}^n$ and $p(\xi) = 0$*

$$d(\Gamma, \operatorname{grad} p(\xi)) + d(\Gamma, -\operatorname{grad} p(\xi)) \geqq c\,|\operatorname{grad} p(\xi)|^{M_1} (1 + |\xi|)^{-M_2} \tag{3.6}$$

where $d(\Gamma, \theta)$ denotes the distance from θ to Γ.

PROOF. For fixed $R > 0$ we denote by $f_R(t)$ the infimum of the left-hand side in (3.6) when $|\xi| < R$, $p(\xi) = 0$, $|\operatorname{grad} p(\xi)| > t$. This is a positive increasing piecewise algebraic function of $t > 0$, by the hypothesis and the Tarski-Seidenberg theorem. Hence

$$f_R(t) > c(R) t^{\gamma(R)}$$

where $c(R) > 0$ and $\gamma(R)$ is a positive rational number which comes from the Puiseux series expansion of an algebraic function of degree independent of R. Hence $\gamma(R) \leqq M$ for some M. Another application of the Tarski-Seidenberg theorem to the infimum of

$$(d(\Gamma, \operatorname{grad} p(\xi)) + d(\Gamma, -\operatorname{grad} p(\xi)))\,|\operatorname{grad} p(\xi)|^{-M}$$

when $|\xi| < R$ and $p(\xi) = 0$ will now prove (3.6).

PROOF OF THEOREM 3.6. We choose the coordinates so that p is non-characteristic with respect to ξ_n. For the complex zeros of p we have then

$$|\xi_n| \leqq C(1 + |\xi'|). \tag{3.7}$$

If $R(\xi')$ is the discriminant of p as a polynomial in ξ_n and $p'_n = \partial p/\partial\xi_n$, it follows that

$$|R(\xi')| \leqq C|p'_n(\xi)|(1 + |\xi'|)^M \text{ when } p(\xi) = 0. \tag{3.8}$$

The zeros $\xi_n = \tau(\xi')$ of p are analytic in the set where $R(\xi') \neq 0$ and they satisfy estimates of the form

$$|D^\alpha\tau(\xi')| < C|R(\xi')|^{-M_1}(1 + |\xi'|)^{M_2}. \tag{3.9}$$

For proofs of these classical elementary facts see, e.g., [4, Lemmas A.3 and A.4].

We shall define u by the integral

$$u(x) = \int \exp(i\langle x,\xi\rangle)R(\xi')^\sigma\hat{g}(\xi)/p(\xi)d\xi \tag{3.10}$$

where $g \in C_0^\infty(\mathbb{R}^n)$, $\hat{g}$ is not zero at every simple real zero of p and σ is a large positive integer. The precise definition of the integrand as a distribution at the zeros of p will be given below.

We shall consider separately the contributions to (3.10) for ξ' in different components Ω of $\{\xi' \in \mathbb{R}^{n-1};\ R(\xi') \neq 0\}$. In view of the implicit function theorem the number of real zeros τ of $p(\xi',\tau)$ is constant when $\xi' \in \Omega$. We denote them by $\tau_1(\xi') < \cdots < \tau_\mu(\xi')$. (Possibly there may be no zero in which case much that follows is trivial.) For $\xi' \in \Omega$ we have by the Lagrange interpolation formula

$$\hat{g}(\xi)R(\xi')^\sigma/p(\xi) = \sum_1^\mu a_{\nu,\Omega}(\xi')/(\xi_n - \tau_\nu(\xi')) + b_\Omega(\xi) \tag{3.11}$$

where $a_{\nu,\Omega}(\xi') = \hat{g}(\xi',\tau_\nu(\xi'))R(\xi')^\sigma/p'_n(\xi',\tau_\nu(\xi'))$. It is clear that $a_{\nu,\Omega} \in C^\infty(\Omega)$ and that $b_\Omega \in C^\infty(\Omega \times \mathbb{R})$.

We choose a C^∞ function ψ on $\mathbb{R}$ so that $\psi(t) = 0$ for $t > 0$ and $\psi(t) = -1$ for $t < -1$. Then we have

$$i\hat{\psi}(\tau) - (\tau + i0)^{-1} \in C^\infty$$

where $(\tau + i0)^{-1} = \lim_{\varepsilon\to+0}(\tau + i\varepsilon)^{-1}$ in $\mathscr{D}'$. Furthermore, $\hat{\psi}$ is rapidly decreasing at infinity. To improve the behavior as $\xi_n \to \infty$ we modify the decomposition (3.11) to

$$(3.11)' \quad \hat{g}(\xi)R(\xi')^\sigma/p(\xi) = \sum_1^\mu a_{\nu,\Omega}(\xi')(\pm i)\hat{\psi}(\pm(\xi_n - \tau_\nu(\xi'))) + b'_\Omega(\xi).$$

At the same time (3.11)′ defines the left-hand side as a distribution of ξ_n for fixed ξ' as soon as we choose the signs. Comparison with (3.11) gives if we consider separately the cases when ξ_n is much larger than or comparable to $|\xi'|$ that for any M, ρ, α

$$(3.12) \qquad |D^\alpha a_{\nu,\Omega}|(1 + |\xi'|)^M + |D^\alpha b'_\Omega|(1 + |\xi|)^M \leqq C|R(\xi')|^\rho$$

provided that σ is larger than some number depending only on α, ρ. When $\rho > 0$ this means in particular that the left-hand side vanishes on the boundary of Ω.

By hypothesis

$$(3.13) \qquad \pm(-\tau'_\nu(\xi'), 1) = \pm \operatorname{grad} p(\xi)/p'_n(\xi) \in \Gamma$$

for exactly one choice of the sign. Since Γ is closed and Ω is connected it follows that the sign for which (3.13) is valid depends only on ν and not on ξ'. We choose this sign in (3.11)′ and in the following final form of (3.10)

$$(3.14) \quad u(x) = \sum_\Omega \int_\Omega d\xi \int \left(\sum_1^\mu a_{\nu,\Omega}(\xi')(\pm i)\hat{\psi}(\pm(\xi_n - \tau_\nu(\xi'))) + b'_\Omega(\xi) \right) \exp(i\langle x, \xi\rangle) d\xi_n$$

$$= \sum_\Omega \sum_\nu (\pm 2\pi i)\psi(\pm x_n) I_{\nu,\Omega}(x) + \sum_\Omega \int b'_\Omega(\xi) \exp(i\langle x, \xi\rangle) d\xi.$$

Here

$$I_{\nu,\Omega}(x) = \int_\Omega \exp(i(\langle x', \xi'\rangle + x_n \tau_\nu(\xi'))) a_{\nu,\Omega}(\xi') d\xi' .$$

We may differentiate under the integral sign in (3.14) and this gives

$$p(D)u(x) = \int \exp(i\langle x, \xi\rangle)\hat{g}(\xi)R(\xi')^\sigma d\xi = (2\pi)^n R(D')^\sigma g \in C_0^\infty.$$

Since $\hat{u} = (2\pi)^n \hat{g} R^\sigma/p$ when $p \neq 0$ and this is not an entire function we conclude that u does not have compact support.

If σ is large enough the functions b'_Ω in the different components Ω combine to a function $b' \in C^N(\mathbb{R}^n)$ which is rapidly decreasing as well as its derivatives of order $\leqq N$. Hence the corresponding contribution to u is in C^∞ and all its derivatives are $O(|x|^{-N})$ as $x \to \infty$.

It remains to estimate the terms in the first sum in (3.14). Assume that (3.13) is valid with the plus sign. Since $\psi(x_n) = 0$ when $x_n \geqq 0$ the corresponding term is 0 unless $x_n < 0$. If

$$\Gamma^- = \{x \in \Gamma;\, x_n < 0\}$$

we have for some constants C, N', N''

$$|x| \leqq C|x' + x_n\tau_v'(\xi')||R(\xi')|^{-N'}(1 + |\xi'|)^{N''}, x \in \Gamma^-, \xi' \in \Omega. \tag{3.15}$$

In view of the homogeneity it suffices to prove this when $x_n = -1$. Then we can apply (3.6) with $\xi = (\xi', \tau_v(\xi'))$ noting that $(-\tau_v'(\xi'), 1) \in \Gamma$ and that

$$\operatorname{grad} p(\xi) = p_n'(\xi)(-\tau_v'(\xi'), 1).$$

It follows that

$$c|p_n'(\xi)|^{M_1 - 1}(1 + |\xi|)^{-M_2} \leqq |x - (\tau_v'(\xi'), -1)| = |x' - \tau_v'(\xi')|$$

which gives (3.15) with $|x_n|$ instead of $|x|$ in the left-hand side. Since

$$|x'| \leqq |x' + x_n\tau_v'(\xi')| + |x_n\tau_v'(\xi')|$$

we conclude in view of (3.9) that (3.15) is valid.

We shall now integrate by parts in $I_{v,\Omega}$ using the fact that

$$L\exp(i(\langle x', \xi'\rangle + x_n\tau_v(\xi'))) = \exp(i(\langle x', \xi'\rangle + x_n\tau_v(\xi'))$$

if $L = -i|x' + x_n\tau_v'|^{-2} \sum_1^{n-1} (x_j + x_n\partial\tau_v/\partial\xi_j)\partial/\partial x_j$. If σ is large enough it follows that

$$I_{v,\Omega}(x) = \int_\Omega \exp(i(\langle x', \xi'\rangle + x_n\tau_v(\xi')))({}^tL)^{N+1}a_{v,\Omega}(\xi')d\xi$$

where ${}^tL$ is the adjoint of L. For sufficiently large σ the estimates (3.9), (3.12) and (3.15) give

$$|I_{v,\Omega}(x)| \leqq C|x|^{-N-1}\int_\Omega (1 + |\xi|)^{-n-1}d\xi,\ x \in \Gamma^-,$$

where C is independent of Ω and v. Hence

$$u(x) = O(|x|^{-N}),\ x \to \infty \text{ in } \Gamma,$$

which completes the proof of Theorem 3.6.

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LUNDS UNIVERSITET
MATHEMATISKA INSTITUTIONEN
BOX 725
LUND, SWEDEN